① Aussage: $\forall n \in \mathbb{N}: \sum_{k=1}^{2^n-1} \frac{1}{k} \geq \frac{n}{2}$

Beweis durch Induktion:

IA: $n=1$: $\sum_{k=1}^{1} \frac{1}{k} \geq \frac{n}{2}$ $\quad \frac{1}{1} \geq \frac{1}{2}$ ✓

IS: $\sum_{k=1}^{2^{n+1}-1} \frac{1}{k} = \sum_{k=1}^{2^n-1} \frac{1}{k} + \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k}$

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k}$$

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=0}^{2^n-1} \frac{1}{2^n+k}$$

$S_n = \sum_{k=0}^{2^n-1} \frac{1}{2^n+k}$ $\qquad 0 \leq k \leq 2^n - 1$

$0 < k + 2^n \leq 2^n + 2^n - 1$

$0 < k + 2^n \leq 2^{n+1} - 1$

$$\frac{1}{k+2^n} \geq \frac{1}{2^{n+1}-1}$$

$$S_n \geq 2^n \cdot \left(\frac{1}{2^{n+1}-1}\right) = \frac{2^n}{2^n\left(2-\frac{1}{2^n}\right)} = \frac{1}{2-\frac{1}{2^n}}$$

$$S_n \geq \frac{1}{2-\frac{1}{2^n}} > \frac{1}{2}$$

② Aussage: $\forall n \in \mathbb{N}: \forall x \in \mathbb{R},\ x \geq -1: (1+x)^n \geq 1 + nx$

Beweis: IA: $n=1$:

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k}$$

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=0}^{2^n-1} \frac{1}{2^n+k}$$

$$S_n \geq 2 \cdot \left(\frac{}{2^{n+1}-1}\right) = 2^n\left(2-\frac{1}{2^n}\right) = \frac{}{2-\frac{1}{2^n}}$$

$$S_n \geq \frac{1}{2-\frac{1}{2^n}} > \frac{1}{2}$$

② Aussage: $\forall n \in \mathbb{N}\ \forall x \in \mathbb{R},\ x \geq -1:\ (1+x)^n \geq 1+nx$

Beweis: IA: $n=1$: $(1+x)^1 = 1+x \geq 1+1\cdot x$

IS: $(1+x)^{n+1} \geq (1+nx)(1+x) = 1+nx+x+nx^2 \geq 1+nx+x$

$(1+x)^{n+1} \geq 1+nx+x = (1+n)x+1$

IS. $\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \underbrace{\sum_{k=1}^{n} \frac{1}{k(k+1)}}_{\frac{n}{n+1}} + \sum_{k=n+1}^{n+1} \frac{1}{k(k+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} =$

$$\sum_{k=0}^{2^n-1} \frac{1}{2^n+k} > \frac{1}{2} \Rightarrow \sum_{k=1}^{2^{n+1}-1} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=0}^{2^n-1} \frac{1}{2^n+k} > \frac{n}{2} + \frac{1}{2} = \frac{(n+1)}{2}$$

$$\frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{\cancel{(n+1)}(n+2)} = \frac{n+1}{n+2}$$

b) $\forall n \in \mathbb{N}: \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1}$

$$\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = \frac{1}{1} - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

IA. $n=1$: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$

IS. $\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \underbrace{\sum_{k=1}^{n} \frac{1}{k(k+1)}}_{\frac{n}{n+1}} + \sum_{k=n+1}^{n+1} \frac{1}{k(k+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} =$

$$\sum_{k=0}^{2^n-1} \frac{1}{2^n+k} > \frac{1}{2} \Rightarrow \sum_{k=1}^{2^{n+1}-1} \frac{1}{k} \geq \frac{n}{2} + \sum_{k=0}^{2^n-1} \frac{1}{2^n+k} > \frac{n}{2} + \frac{1}{2} = \frac{(n+1)}{2}$$

Aufgabe 3 sei $n \in \mathbb{N}$ $x_i > 0$ $i \in \{1, \dots, n\}$ mit

$x_1 \cdots x_n = 1$

z.z. $x_1 + \dots + x_n \geq n$

(IA) $n = 1$ $x_1 = 1 \geq 1$ ✓

(IV)

(IS) $n \mapsto n+1$

$x_1 \cdots x_{n+1} = 1$ z.z. $\underline{x_1 + \dots + x_{n+1} \overset{!}{\geq} n+1}$

Fall 1: $\forall x_i = 1,\ i \in \{1, \dots, n\}$

(IV)

(IS) $n \mapsto n+1$

$x_1 \cdots x_{n+1} = 1$ z.z. $\underline{x_1 + \dots + x_{n+1} \overset{!}{\geq} n+1}$

Fall 1: $\forall x_i = 1,\ i \in \{1, \dots, n\}$

$x_1 + \dots + x_{n+1} = 1 + 1 + \dots + 1 = n+1$

Fall 2: $\exists x_i \neq 1,\ i \in \{1 \dots n\}$

$x_j > 1,\ x_k < 1,\ j, k \in \{1 \dots n\},\ j \neq k$

$x_1 \cdots (x_j x_k) \cdots x_{n+1} = 1 \overset{(IV)}{\Rightarrow} x_1 + \dots + (x_j x_k) + \dots + x_{n+1} \geq n$

$x_j + x_k$. $\quad x_j \cdot x_k$

wegen $x_j > 1$, $x_k < 1$

$$\underbrace{(x_j - 1)}_{>0}\underbrace{(x_k - 1)}_{<0} < 0$$

$$\Rightarrow x_j x_k - x_j - x_k + 1 < 0$$

$$\Rightarrow x_j + x_k > x_j x_k + 1$$

$$\Rightarrow \underline{x_1 + \dots + x_j + \dots + x_k + x_{n+1}} > \overbrace{x_j + \dots + (x_j x_k) + \dots + x_{n+1}}^{\geq n} + 1 \geq \underline{n+1}$$

$$\Rightarrow x_1 + \dots + x_{n+1} \geq n+1$$

$\square$